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Convex and Strongly Convex Fuzzy Sets

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Properties of the support and of the core of convex and strongly convex fuzzy sets are considered. The convex and strongly convex fuzzy sets in the real line are characterized by means of the piece-wise monotonic functions. © 1987 Academic Press, Inc.

1. INTRODUCTION

A concept of a convex fuzzy set was introduced by Zadeh [11] in the first paper on fuzzy sets. We rewrite the Zadeh's definition using a simpler notation.

Let $A\colon \mathbb{R}^n \to [0, 1]$ denote a fuzzy set in $\mathbb{R}^n$ for a given positive integer n.

DEFINITION 1. A fuzzy set A is convex if

$$A(tx+(1-t)y) \geqslant \min(A(x), A(y)) \qquad \text{for} \quad x, y \in \mathbb{R}^n, t \in (0, 1). \tag{1}$$

A is strongly convex if

$$\begin{aligned} &A(tx+(1-t)y) \\ &\quad > \min(A(x), A(y)) \qquad \text{for} \quad x \neq y, x, y \in \mathbb{R}^n, t \in (0, 1). \end{aligned} \tag{2}$$

Note that any strongly convex fuzzy set is convex. Different properties of convex fuzzy sets are described by Chang [3], Katsaras and Liu [8], and Lowen [9]. However, we do not know any deeper considerations of strongly convex fuzzy sets.

Our considerations are stimulated by the papers of Dubois and Prade [5] and [6] because of the differences between the two definitions of fuzzy number used there.

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2. Support of Convex Fuzzy Set

After Bellman and Zadeh [1] we put

Definition 2. A support of a fuzzy set A is the set

$$S(A) = \{x \in \mathbb{R}^n \mid A(x) > 0\}. \tag{3}$$

It is obvious that $S(A) = \varnothing$ iff $A = 0$, i.e.,

$$A(x) = 0 \qquad \text{for} \quad x \in \mathbb{R}^n.$$

Thus the assumption $A \neq 0$ is equivalent to $S(A) \neq \varnothing$.

Theorem 1. (a) *If A is a convex fuzzy set then $S(A)$ is a convex set.*

(b) *If A is a strongly convex fuzzy set then $S(A) = \mathbb{R}^n$.*

Proof. The first property is implied directly by (1) and (3).

If A is a strongly convex fuzzy set, then from (2) for any $x, y \in \mathbb{R}^n$, $y \neq 0$ and $t = \frac{1}{2}$. We obtain

$$A(x) = A(\tfrac{1}{2}(x-y) + \tfrac{1}{2}(x+y)) > \min(A(x-y), A(x+y)) \geqslant 0,$$

i.e., $x \in S(A)$, which proves (b).

3. Core of Convex Fuzzy Set

Using notion of the height of a fuzzy set (cf. Zadeh [12]) we put

$$h(A) = \sup_{x \in \mathbb{R}^n} A(x). \tag{4}$$

The following definition is an equivalent form of that introduced by Zadeh [11]:

Definition 3. A $q \in \mathbb{R}^n$ is a core point of the fuzzy set A if there exists a sequence (q_k) such that

$$q_k \to q, \qquad A(q_k) \to h(A) \qquad \text{for} \quad k \to \infty. \tag{5}$$

The set of all core points of A is named the core of A and it is denoted by $C(A)$.

Remark 1. If $A \neq 0$ then the sequence (q_k) in (5) can be choosen from the support $S(A)$ and therefore $C(A) \subset \operatorname{Cl} S(A)$.

PROPOSITION 1. *If $C(A)=\varnothing$ then there exists a sequence (x_k) such that*

$$A(x_k)\neq h(A) \qquad \textit{for} \quad k=1,2,..., \tag{6}$$

$$|x_k|\to\infty \qquad \textit{for} \quad k\to\infty, \tag{7}$$

$$A(x_k)\to h(A) \qquad \textit{for} \quad k\to\infty. \tag{8}$$

Proof. From (4) we get the existence of a sequence (x_k) with property (8). If this sequence is bounded in $\mathbb{R}^n$, then we obtain a subsequence (q_k) convergent to certain $q\in C(A)$, which makes a contradiction with the assumption $C(A)=\varnothing$. Thus our sequence (x_k) is nonbounded and can be reduced to subsequence with property (7). Finally, property (6) is obvious under assumption $C(A)=\varnothing$.

From the above proof and property (5) we get

COROLLARY 1. *$C(A)=\varnothing$ iff any sequence with property* (8) *is nonbounded.*

COROLLARY 2. *If A is a convex fuzzy set in $\mathbb{R}$ (case $n=1$), then $C(A)=\varnothing$ iff there exists a sequence (x_k) with properties* (6)–(8).

Proof. After Corollary 1 it suffices to prove that (6)–(8) are contradictory to (5). Let us assume the existence of both sequences (q_k) and (x_k) in (5)–(8). In the case $n=1$, these sequences can be assumed monotonic and with strictly increasing sequence $(A(x_k))$ (cf. (6)).

If $x_k\to+\infty$ for $k\to\infty$, then for suitable great indices j, k we get

$$q_j<x_k<x_{k+1}, \qquad A(x_k)<\min(A(q_j), A(x_{k+1}))$$

in contradiction to (1). Similarly we get a contradiction if $x_k\to-\infty$ for $k\to\infty$. Therefore the existence of a sequence (x_k) with properties (6)–(8) implies that any sequence with property (8) is nonbounded. It finishes the proof.

COROLLARY 3. *If $S(A)$ is bounded then $C(A)\neq\varnothing$.*

Proof. For $A=0$ it is obvious. If $A\neq 0$ then in $S(A)$ there exists a sequence with property (8). This sequence is bounded together with $S(A)$ and $C(A)\neq\varnothing$ by Corollary 1.

THEOREM 2. *The core of a convex fuzzy set is a closed convex set.*

Proof. Let A be a convex fuzzy set. The case $C(A)=\varnothing$ is obvious. If $C(A)\neq\varnothing$ and c is an accumulation point of $C(A)$, then in $C(A)$ there exists a sequence (c_i) convergent to c. From Definition 3 we obtain a family of sequences (q_k^i) such that

$$q_k^i \to c_i, \qquad A(q_k^i) \to h(A) \quad \text{for} \quad k \to \infty.$$

Thus we get

$$q_k^k \to c, \qquad A(q_k^k) \to h(A) \quad \text{for} \quad k \to \infty,$$

i.e., $c \in C(A)$ and therefore $C(A)$ is a closed set.

If $C(A) \neq \varnothing$ and

$$q = tx + (1-t)\,y \qquad \text{for given} \quad x, y \in C(A), t \in (0, 1)$$

then there exist sequences (x_k), (y_k) such that (Definition 3)

$$x_k \to x, \quad y_k \to y, \quad A(x_k) \to h(A), \quad A(y_k) \to h(A) \quad \text{for } k \to \infty.$$

Using (1) we see that the sequence

$$q_k = tx_k + (1-t)\,y_k \qquad \text{for} \quad k = 1, 2, ...,$$

fulfils condition

$$h(A) \geqslant A(q_k) \geqslant \min(A(x_k), A(y_k)) \qquad \text{for} \quad k = 1, 2, ...,$$

and we obtain (5), i.e., $q \in C(A)$. This proves the convexity of $C(A)$.

Remark 2. The convexity of the core of a fuzzy set was proved by Zadeh [11] in another way. Our proof is connected with a new formulation of Definition 3.

Now we rewrite (cf. Lowen [9, Proposition 6.1])

LEMMA 1. *A is a convex fuzzy set iff for any positive integer k we have*

$$A\left(\sum_{i=1}^{k} t_i x_i\right) \geqslant \min_{1 \leqslant i \leqslant k} A(x_i) \qquad \textit{for} \quad x_i \in \mathbb{R}^n,$$

$$t_i \in [0, 1], i = 1, ..., k, \sum_{i=1}^{k} t_i = 1. \tag{9}$$

Using this lemma we prove

THEOREM 3. *For any convex fuzzy set A we have*

$$A(q) = h(A) \qquad \textit{for} \quad q \in \operatorname{Int} C(A). \tag{10}$$

Proof. Let $q \in \operatorname{Int} C(A)$. There exists a constant $d > 0$ such that $C(A)$ contains the n-dimensional cube with the centre in q and with edges of the length $3\,d$. The vertices of this cube we denote by $x^1, \ldots, x^{2^n} \in C(A)$. Choosing $q^1, \ldots, q^{2^n} \in C(A)$ such that

$$|q^i - x^i| < d \qquad \text{for} \quad i = 1, \ldots, 2^n,$$

we see that the polyhedron with the vertices $q^1, \ldots, q^{2^n}$ contains q. Therefore q is a convex combination of this vertices, i.e., there exist $t_1, \ldots, t_{2^n} \in [0, 1]$ such that

$$\sum_{i=1}^{2^n} t_i = 1, \qquad q = \sum_{i=1}^{2^n} t_i x^i.$$

Now, using (9) for $k = 2^n$ we obtain the inequality

$$A(q) \geqslant \min_{1 \leqslant i \leqslant 2^n} A(q^i).$$

Moreover (cf. Definition 3), there exist sequences (q_k^i), $i = 1, \ldots, 2^n$, such that

$$|q_k^i - x^i| < d, \qquad q_k^i \to q^i,$$
$$A(q_k^i) \to h(A) \quad \text{for} \quad k \to \infty,\, i = 1, \ldots, 2^n$$

and the above argument leads us to the inequalities

$$A(q) \geqslant \min_{1 \leqslant i \leqslant 2^n} A(q_k^i) \qquad \text{for} \quad k = 1, 2, \ldots.$$

Therefore

$$A(q) \geqslant \sup_k \min_{1 \leqslant i \leqslant 2^n} A(q_k^i) = h(A)$$

which proves (10).

Immediately from (2) and Theorem 3 we get

COROLLARY 4. *If A is a strongly convex fuzzy set then* $\operatorname{Int} C(A) = \varnothing$.

Note that the relative interior of $C(A)$ in Corollary 4 has the dimension not greater than $n-1$, and it is equal to the dimension of $C(A)$ (cf. Rockafellar [10, Theorem 6.2]). Thus using Theorem 2 we get

THEOREM 4. *The core of a strongly convex fuzzy set is a closed convex set of the dimension not greater than $n-1$.*

COROLLARY 5 (cf. Zadeh [11]). *If A is a strongly convex fuzzy set in $\mathbb{R}$ (case $n = 1$), then $C(A) = \varnothing$ or $C(A)$ is a singleton (the unique core point).*

4. Characterization of Convex Fuzzy Sets in $\mathbb{R}$

As a direct consequence of Definition 1 we state

PROPOSITION 2. *A is a (strongly) convex fuzzy set in $\mathbb{R}^n$ iff its restriction to any stright line in $\mathbb{R}^n$ is (strongly) convex.*

This equivalence allows us to reduce a detailed consideration of convex fuzzy sets to the case $n = 1$. So from now on we consider a fuzzy set $A: \mathbb{R} \to [0, 1]$. In this case we see that (cf. Definition 1).

LEMMA 2. *A is a convex fuzzy set in $\mathbb{R}$ iff*

$$(x < z < y) \Rightarrow (A(z) \geqslant \min(A(x), A(y))) \quad \text{for} \quad x, y, z \in \mathbb{R}. \tag{11}$$

A is a strongly convex fuzzy set in $\mathbb{R}$ iff

$$(x < z < y) \Rightarrow (A(z) > \min(A(x), A(y))) \quad \text{for} \quad x, y, z \in \mathbb{R}. \tag{12}$$

Let us denote

$$\begin{aligned} A_-(z) &= \sup_{x<z} A(x), \\ A_+(z) &= \sup_{y>z} A(y) \qquad \text{for} \quad x, y \in \mathbb{R}, z \in \bar{\mathbb{R}}. \end{aligned} \tag{13}$$

Remark 3. Function $A_-: \bar{\mathbb{R}} \to [0, 1]$ is increasing, function $A_+: \bar{\mathbb{R}} \to [0, 1]$ is decreasing and both are lower semicontinuous.

LEMMA 3. *A is a convex fuzzy set in $\mathbb{R}$ iff*

$$A(z) \geqslant \min(A_-(z), A_+(z)) \quad \text{for} \quad z \in \mathbb{R}. \tag{14}$$

Proof. Using the notation (13) we see that (11) and (14) are equivalent as both equivalent to the condition

$$A(z) \geqslant \sup_{x<z} \sup_{y>z} \min(A(x), A(y)) \quad \text{for} \quad x, y, z \in \mathbb{R}.$$

THEOREM 5. *$A \neq 0$ is a convex fuzzy set in $\mathbb{R}$ iff there exist constants p, q, r, $s \in \bar{\mathbb{R}}$ such that*

$$-\infty \leqslant p \leqslant q \leqslant r \leqslant s \leqslant +\infty, \tag{15}$$

$$A(x) = 0 \quad \text{for} \quad x < p, \tag{16}$$

$$A(x) \leqslant A(y) \quad \text{for} \quad x < y,\ x, y \in [p, q), \tag{17}$$

$$A(q) \geqslant \min(A_-(q), A_+(q)), \tag{18}$$

$$A(x) = h(A) \qquad \textit{for} \quad x \in (q, r), \tag{19}$$

$$A(r) \geqslant \min(A_-(r), A_+(r)), \tag{20}$$

$$A(x) \geqslant A(y) \qquad \textit{for} \quad x < y,\, x,\, y \in (r, s], \tag{21}$$

$$A(x) = 0 \qquad \textit{for } x > s, \tag{22}$$

where x, $y \in \mathbb{R}$ (*values* $A(-\infty)$ *and* $A(+\infty)$ *possible in* (18) *or* (20) *are meant as adequate limits* (8)).

Proof. (I) If A is a convex fuzzy set in $\mathbb{R}$ then its support and core are intervals (cf. Theorems 1 and 2), and $C(A) \subset \mathrm{Cl}\, S(A)$ (cf. Remark 1). Putting

$$[p, s] = \mathrm{Cl}\, S(A), \tag{23}$$

$$q = r = -\infty \qquad \text{or} \qquad q = r = +\infty \qquad \text{or} \qquad [q, r] = C(A) \tag{24}$$

according to Corollary 2 and Theorem 2, we get the suitable constants (15). Now (16) and (22) are implied by (23), (19) is a special case of (10) in Theorem 3, and (18) and (20) are special cases of (14) in Lemma 3. We prove (17).

If (q_k) denotes the sequence (5) and x, $y \in [p, q)$, $x < y$, then for suitable great k (e.g., $k > m$) we have $x < y < q_k$ and therefore (cf. (11))

$$A(y) \geqslant \sup_{k > m} \min(A(x), A(q_k)) = \min(A(x), h(A)) = A(x)$$

which proves (17) and similarly we get (21). Thus we obtain (15)–(22).

(II) Now let us assume that a fuzzy set $A \neq 0$ fulfils conditions (16)–(22) for given p, q, r, s in (15). For arbitrary x, y, $z \in \mathbb{R}$, $x < z < y$ we consider the following cases:

(a) if $z < q$ then by (16) or (17) $A(z) \geqslant A(x)$ and we get

$$A(z) \geqslant \min\, (A(x), A(y)); \tag{25}$$

(b) if $z = q$ then (25) is implied by (18);

(c) if $z \in (q, r)$ then (25) is implied by (19);

(d) if $z = r$ then (25) is implied by (20);

(e) if $z > r$ then $A(z) \geqslant A(y)$ by (21) or (22) and we also get (25).

In all possible cases we get (25) and therefore A is convex by virtue of Lemma 2.

We also have (cf. the definition of fuzzy interval by Burdzy and Kiszka [2])

THEOREM 6. *A is a convex fuzzy set in $\mathbb{R}$ iff there exists $q \in \bar{\mathbb{R}}$ such that A fulfils* (18), *A is increasing in $(-\infty, q)$ and A is decreasing in $(q, +\infty)$.*

Proof. (I) If A is a convex fuzzy set then q defined by (24) is suitable on account of Theorem 5.

(II) The second part of the proof is a special case of the argument used in the Proof of Theorem 5.

From the property of monotonic real functions we get

COROLLARY 6. *If A is a convex fuzzy set in $\mathbb{R}$, then it has at most the countable number of discontinuity points.*

After Theorem 6 and Remark 3 we also have

COROLLARY 7. *If A is a lower semicontinuous function then it is a convex fuzzy set iff there exists $q \in \bar{\mathbb{R}}$ such that A fulfils* (18) *and*

$$A(x) = \begin{cases} A_-(x) & \text{for} \quad x < q, \\ A_+(x) & \text{for} \quad x > q, \quad x \in \mathbb{R}. \end{cases} \tag{26}$$

In the case of strongly convex fuzzy set A, it is not constant in any interval (cf. (12)) and similarly as in Theorem 6 we get

THEOREM 7. *A is a strongly convex fuzzy set in $\mathbb{R}$ iff there exists $q \in \bar{\mathbb{R}}$ such that A fulfils* (18) *and A is strictly increasing in $(-\infty, q)$, and A is strictly decreasing in $(q, +\infty)$.*

Note that condition (18) can be omitted in Theorems 5–7 if A is continuous. Then we also get monotonicity in intervals $(-\infty, q]$ and $[q, +\infty)$.

5. CONCLUSION

The results are useful in the consideration of fuzzy intervals as convex fuzzy subsets in $\mathbb{R}$ (cf. Drewniak [4], Dubois and Prade [5]). In the case of continuous A we get the representation (26) used by Dubois and Prade [6] in bounded intervals with monotone and strictly monotone A_- and A_+. However, Goeschel and Voxman [7] show that the most useful definition of fuzzy number or fuzzy interval requires upper semicontinuity of A (only) while A_- and A_+ are lower semicontinuous.

Zadeh [11] has also defined the third kind of convexity—the strict one but the strict convexity must be examined by other methods (in $\mathbb{R}$ it reduces to convexity).

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